Logistic Regression

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Classification

- Recall that we we're trying to predict continuous values using regression.

- If we're trying to predict the values *y* which only take on a small amount of discrete values, it is called classification.

- For now we will focus on binary classification, ie, predicting either a **0** or a **1**.

- **0** will be called the negative class, and
  1 will be called the positive class.

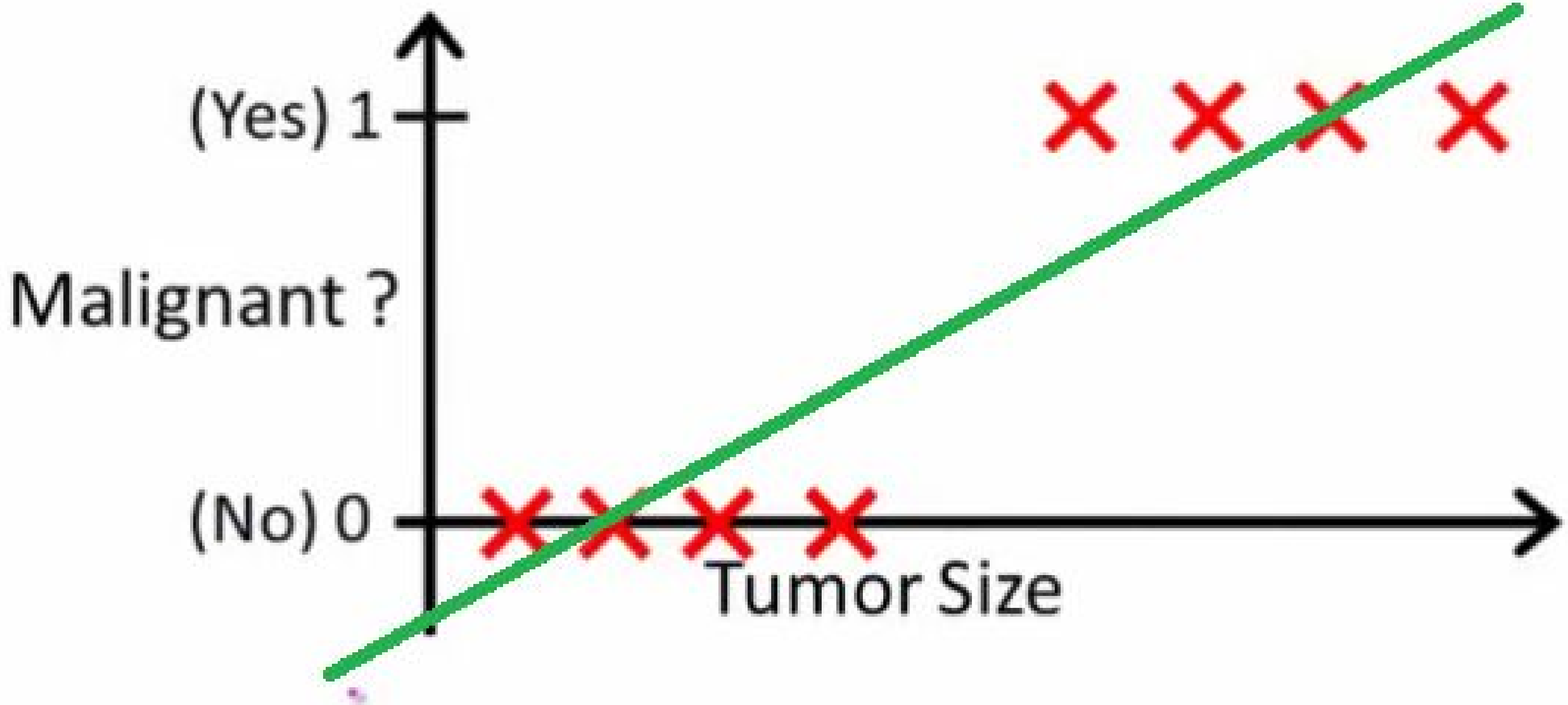
Logistic Regression

- We could attempt to tackle this classification problem with the linear regression algorithm.
- However, it is easy to construct an example where this performs poorly, as we will see on the next slide.
- For initial intuition, it does not make sense for the hypothesis function to output values greater than 1 or less than 0 when $y \in \{0, 1\}$.

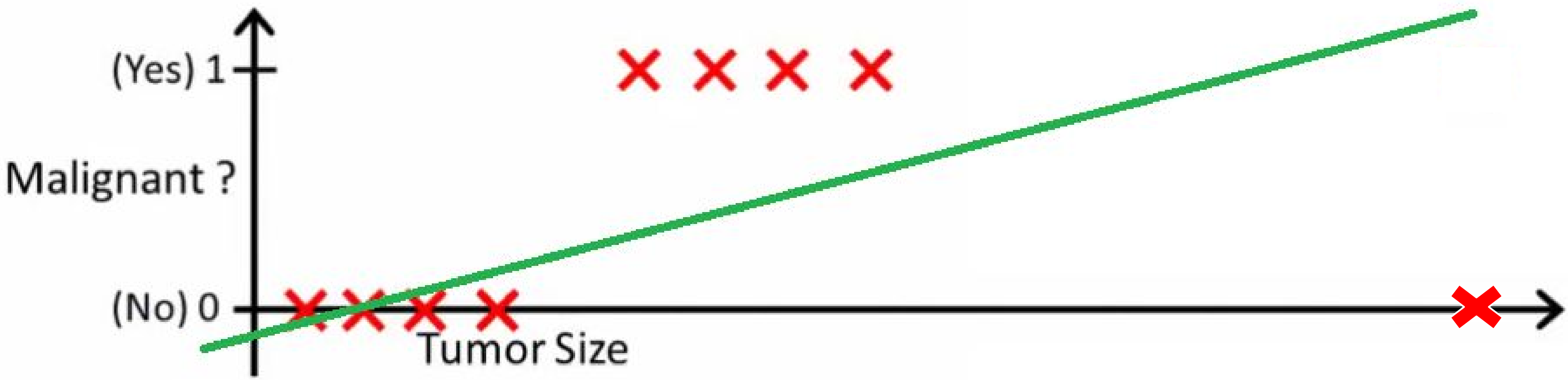
Linear Regression Binary Classification Example

- Suppose we are trying to predict whether a tumor is malignant based on its size.
- Malignant tumors are labeled **1**, and benign tumors are labeled **0**.
- To make predictions using linear regression, we could say if *h(x)* outputs a value larger than 0.5, predict malignant, otherwise predict benign.

Example Cont.



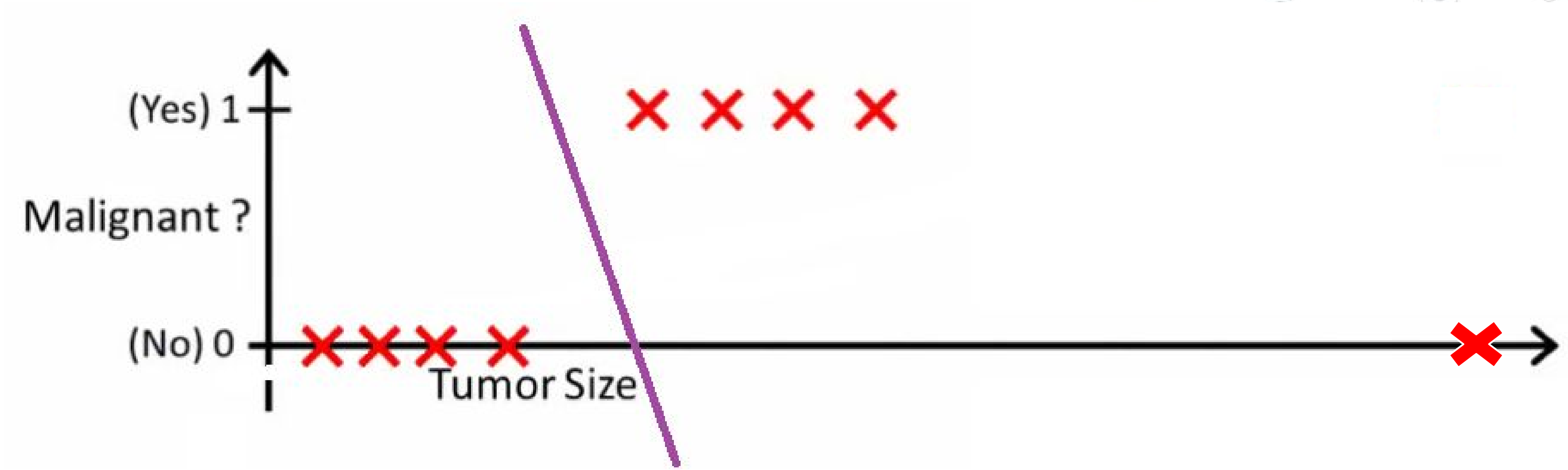
Example Cont.



- Now notice that $h(x) > 0.5 \Rightarrow$ malignant does not work anymore. We would have to alter our *h*.

- But we can't just change *h* every time a new sample arrives - it should be fixed after training.

Example Cont.



- Both linear and logistic predict straight lines.
- Linear interpolates the output and predicts the value for *x* we haven’t seen.
- Logistic says all points sitting to the right of the classifier line belong to one class, and the left belong to the other.
  - In this case, *h(x)* represents the probability that *x* belongs to the positive class.

Sigmoid Function

- We will change the form of $h_\theta(x)$:

$$h_\theta(x) = g(\theta^T x) = \frac{1}{1 + e^{-\theta^T x}},$$

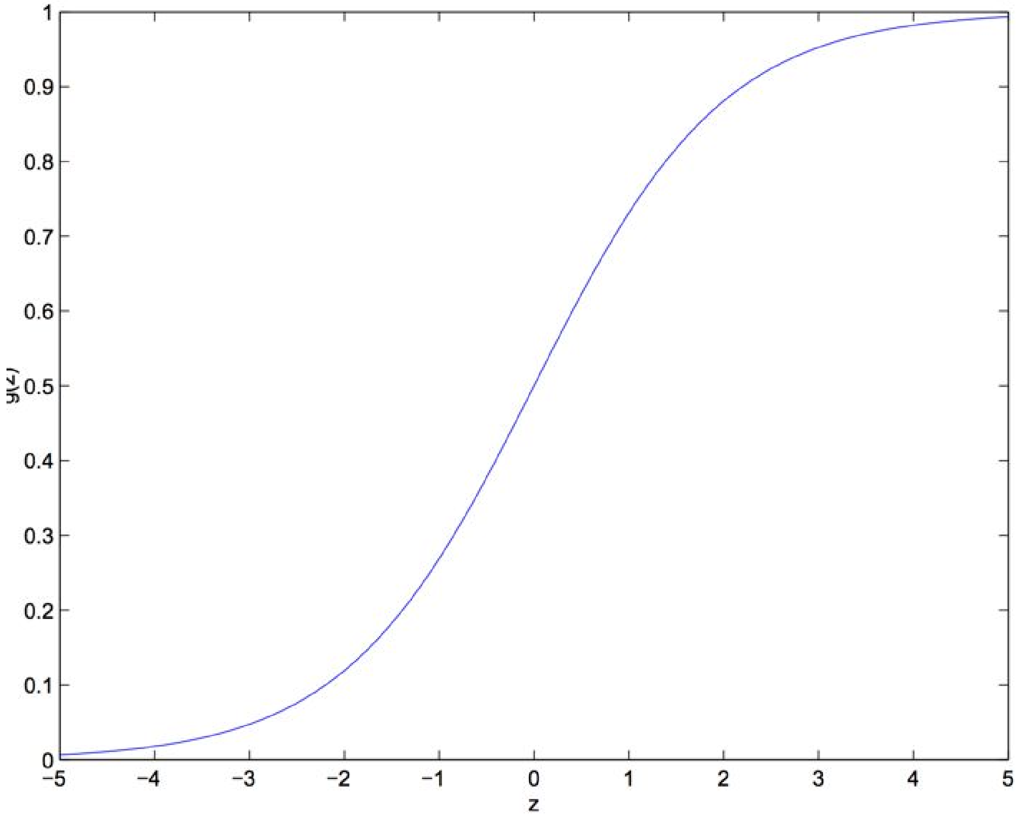
where

$$g(z) = \frac{1}{1 + e^{-z}}$$

is called the logistic or sigmoid function.

Sigmoid Function

- Here is a plot of *g(z)*: (visualize scaling)



$$\lim_{z \to \infty} g(z) = 1, \qquad \lim_{z \to -\infty} g(z) = 0$$

Sigmoid Function

- So we kind of arbitrarily chose this g due to the fact that it increases from 0 to 1.

- In fact, there are many reasons why we use the logistic function, the first being its *smoothness* and easily computable derivative:

$$
\begin{aligned}
g'(z) &= \frac{d}{dz}\frac{1}{1+e^{-z}} \\
&= \frac{1}{(1+e^{-z})^2}\left(e^{-z}\right) \\
&= \frac{1}{(1+e^{-z})}\cdot\left(1-\frac{1}{(1+e^{-z})}\right) \\
&= g(z)(1-g(z)).
\end{aligned}
$$

Fitting the Logistic Regression Model

- Similar to linear regression, we want to find θ to *best* fit our data for future predictions.
- Let's endow the classification problem with some probabilistic assumptions (as we did with linear regression), and then fit the parameters through maximum likelihood!

Fitting the Logistic Regression Model

- Assume that

$$
\begin{aligned}
P(y=1 \mid x;\theta) &= h_\theta(x) \\
P(y=0 \mid x;\theta) &= 1 - h_\theta(x)
\end{aligned}
$$

- Or more compactly, that

$$
p(y|x;\theta) = (h_\theta(x))^y (1 - h_\theta(x))^{1-y}
$$

Fitting the Logistic Regression Model

- Assuming the *m* training examples were generated independently, the likelihood of the parameters is

$$
\begin{aligned}
L(\theta) &= p(\vec{y} \mid X; \theta) \\
&= \prod_{i=1}^{m} p(y^{(i)} \mid x^{(i)}; \theta) \\
&= \prod_{i=1}^{m} \left(h_\theta(x^{(i)})\right)^{y^{(i)}} \left(1 - h_\theta(x^{(i)})\right)^{1-y^{(i)}}
\end{aligned}
$$

Fitting the Logistic Regression Model

- Maximizing the log likelihood will again be easier:

$$\begin{aligned}\ell(\theta) &= \log L(\theta) \\ &= \sum_{i=1}^{m} y^{(i)} \log h(x^{(i)}) + (1 - y^{(i)}) \log(1 - h(x^{(i)}))\end{aligned}$$

- We will maximize this function using gradient descent, thus we need to find its gradient. Let's do it component-wise on a single training example (x,y):

Fitting the Logistic Regression Model

$$h_\theta(x) = g(\theta^T x) = \frac{1}{1 + e^{-\theta^T x}},$$

$$\ell(\theta) = y \log h_\theta(x) + (1 - y) \log(1 - h_\theta(x))$$

Therefore:

$$\begin{aligned}
\frac{\partial}{\partial \theta_j} \ell(\theta) &= \left(y \frac{1}{g(\theta^T x)} - (1 - y) \frac{1}{1 - g(\theta^T x)} \right) \frac{\partial}{\partial \theta_j} g(\theta^T x) \\
&= \left(y \frac{1}{g(\theta^T x)} - (1 - y) \frac{1}{1 - g(\theta^T x)} \right) g(\theta^T x)(1 - g(\theta^T x)) \frac{\partial}{\partial \theta_j} \theta^T x \\
&= \left(y(1 - g(\theta^T x)) - (1 - y) g(\theta^T x) \right) x_j \\
&= (y - h_\theta(x))\, x_j
\end{aligned}$$

since $g'(z) = g(z)(1 - g(z))$.

Fitting the Logistic Regression Model

- This gives us the stochastic gradient *ascent* rule:

$$\theta_j := \theta_j + \alpha(y^{(i)} - h_\theta(x^{(i)}))x_j^{(i)}$$

- This looks identical to the LMS update rule!
- But this is a completely different algorithm.
- Is this a coincidence?
- No!! It is because they are both types of Generalized Linear Models (GLM's).