Linear Algebra and Calculus!

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Eigenvalues and Eigenvectors

- Given a square matrix $A \in \mathbb{R}^{n \times n}$, we say $\lambda \in \mathbb{C}$ is an eigenvalue of A and $x \in \mathbb{C}^n$ is the corresponding eigenvector if

$$Ax = \lambda x, \quad x \neq 0.$$

- Intuitively, this means multiplying A by x results in a new vector in the same direction as x but scaled by λ.

Eigenvalues and Eigenvectors

- Note that if *x* is an eigenvector, then *cx* is an eigenvector for any complex *c*.

- We can rewrite the equation above as

$$(\lambda I - A)x = 0, \quad x \neq 0.$$

- $(\lambda I - A)x = 0$ has a non-zero solution iff $(\lambda I - A)$ has a non-empty nullspace, which only happens if $(\lambda I - A)$ is singular, ie,

$$|(\lambda I - A)| = 0.$$

E-values and E-vectors Properties

- The trace of A is equal to the sum of its eigenvalues,

$$\mathrm{tr} A = \sum_{i=1}^{n} \lambda_i .$$

- The determinant of A is equal to the product of its eigenvalues

$$|A| = \prod_{i=1}^{n} \lambda_i .$$

- The rank of A is equal to the number of non-zero eigenvalues of A.
- The eigenvalues of a diagonal matrix are just the diagonal entries.

Diagonalization

- We can write all the eigenvector equations simultaneously as

$$AX = X\Lambda\,.$$

with $X \in \mathbb{R}^{n\times n}$ the eigenvectors of A and a diagonal matrix Λ whose entries are the eigenvalues A, ie,

$$X \in \mathbb{R}^{n\times n} = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix}, \quad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n).$$

Diagonalization

- If the eigenvectors of A are linearly independent, then X will be invertible, so

$$A = X \Lambda X^{-1}.$$

We say that A is <u>diagonalizable</u>.

Quadratic Forms

- Given any symmetric matrix $A \in \mathbb{R}^{n \times n}$ and vector $x \in \mathbb{R}^n$, the scalar value $x^T A x$ is called a quadratic form.

- Explicitly, we have

$$x^T A x = \sum_{i=1}^{n} x_i (Ax)_i = \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} A_{ij} x_j \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j \; .$$

Definite Matrices

- A is positive definite if for all non-zero vectors $x \in \mathbb{R}^n$

$$x^T A x > 0.$$

- A is negative definite if for all non-zero vectors $x \in \mathbb{R}^n$

$$x^T A x < 0.$$

- Positive and negative definite matrices are full rank and thus invertible.

- For any matrix $A \in \mathbb{R}^{m \times n}$, $A^T A$ is positive semidefinite.

E-values and E-vectors of Symmetric Matrices

- Let $A \in \mathbb{R}^{n\times n}$ be any symmetric matrix:
  - All eigenvalues of A are real.
  - The non-collinear eigenvectors of A are orthonormal.
  - Thus we can decompose A:

$$A = U\Lambda U^T$$

where U is an orthogonal matrix.

E-values and E-vectors of Symmetric Matrices

- We can use this to show that definiteness only depends on sign of eigenvalues:

$$x^T A x = x^T U \Lambda U^T x = y^T \Lambda y = \sum_{i=1}^{n} \lambda_i y_i^2$$

  where $y = U^T x$.
- For any quadratic form $x^T A x$ subject to $\|x\|_2^2 = 1$, its maximum value is the maximum eigenvalue of A, and its minimum value is the minimum eigenvalue of A.

Singular Value Decomposition (SVD)

- Goal: Given any matrix $A \in \mathbb{R}^{m \times n}$, find orthogonal matrices U and V such that

$$A = U \Sigma V^T$$

- If A diagonalizable,

$$A = X \Lambda X^{-1}.$$

- If A positive semidefinite,

$$A = U \Lambda U^T$$

Singular Value Decomposition (SVD)

- See blackboard pictures.
- We know we can find an orthonormal basis for the rowspace of A.
- Can we find one that is mapped into an orthonormal basis for the column space of A?

Singular Value Decomposition (SVD)

- Goal: Find an orthonormal basis $v_1, \ldots, v_r$ for the row space of A such that

$$Av_1 = \sigma_1 u_1, \ldots, Av_r = \sigma_r u_r$$

where $u_1, \ldots, u_r$ is an orthonormal basis for the column space of A.

Singular Value Decomposition (SVD)

- In matrix notation,

$$A\begin{bmatrix} | & & | & | & & | \\ v_1 & \cdots & v_r & v_{r+1} & \cdots & v_m \\ | & & | & | & & | \end{bmatrix} = \begin{bmatrix} | & & | & | & & | \\ u_1 & \cdots & u_r & u_{r+1} & \cdots & u_n \\ | & & | & | & & | \end{bmatrix} \left[\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_r & \\ \hline & 0 & & 0 \end{array}\right]$$

where $v_{r+1}, \ldots, v_m$ orthonormal basis for the null space of A, and $u_{r+1}, \ldots, u_n$ an orthonormal basis for the null space of A^T.

- Or

$$AV = U\Sigma$$

Singular Value Decomposition (SVD)

- But because $\mathcal{N}(A)$ and $\mathcal{R}(A)$ are orthogonal complements, V is orthogonal.
- Similarly, U is orthogonal.
- Therefore

$$AV = U\Sigma \Rightarrow A = U\Sigma V^T$$

Singular Value Decomposition (SVD)

- How do we find U and V?
- Trick:

$$A^T A = V\Sigma^T U^T U \Sigma V^T = V \left[\begin{array}{ccc|c} \sigma_1^2 & & & \\ & \ddots & & 0 \\ & & \sigma_r^2 & \\ \hline & 0 & & 0 \end{array}\right] V^T$$

- Since $A^T A$ is positive semidefinite, V is the orthogonal matrix of eigenvectors of $A^T A$, and its eigenvalues are the squares of the diagonal entries of Σ.

Singular Value Decomposition (SVD)

- Similarly,

$$AA^T = U\Sigma^T V^T V \Sigma U^T = U \left[\begin{array}{ccc|c} \sigma_1^2 & & & \\ & \ddots & & 0 \\ & & \sigma_r^2 & \\ \hline & 0 & & 0 \end{array}\right] U^T$$

- So to find U, simply find the eigenvectors of AA^T.

SVD Example

$$A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$$

Find orthonormal v_1, v_2 in the row space of A ($\mathbb{R}^2$) and orthonormal u_1, u_2 in the column space of A ($\mathbb{R}^2$), and $\sigma_1, \sigma_2 > 0$ such that

$$Av_1 = \sigma_1 u_1, Av_2 = \sigma_2 u_2$$

SVD Example

$$A^T A = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$$

Therefore the eigenvectors of $A^T A$ are

$$\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

with eigenvalues *32* and *18* respectively.

SVD Example

$$AA^T = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}$$

Therefore the eigenvectors of $A^T A$ are

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

again with eigenvalues *32* and *18* respectively (is this surprising?)

SVD Example

Therefore:

$$A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = U\Sigma V^T$$

SVD Example 2

$$A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$$

See blackboard for geometric intuition.

$$A^T A = \begin{bmatrix} 80 & 60 \\ 60 & 45 \end{bmatrix}$$

SVD Example 2

- Therefore the eigenvalues of $A^T A$ are *0* and *125*. Hence:

$$A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} \end{bmatrix} \begin{bmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{4}{5} \end{bmatrix} = U\Sigma V^T$$

So who cares?

- Why is SVD even useful?

- SVD can be used for **dimensionality reduction** - given high dimensional data, one can use SVD to represent the data using less dimensions, while still capturing the most significant (largest eigenvalues) features.

- We will see important applications later.